Dual Optimum Aerodynamic Design for a Conventional Windmill

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The theoretical possibility of designing the blading of a conventional windmill so as to match the optimum loading for maximum output at two different operating conditions (i.e., at two different values of the ratio of tip speed to wind speed) is investigated using the vortex blade element theory. The theory of optimum loading is developed, including the effect of profile drag. For the range of parameters considered, it is shown that the effect of profile drag on blade loading is quite small, although the effect on efficiency is significant. The simpler theory, neglecting profile drag, and first presented by Glauert, then is used to design three sets of blading which match the Glauert loading conditions at two conditions, tip speed ratios of 4 and 5. The first design is for fixed blading and the other two involve pitch changes of 1.5° and 2° between the two conditions, the latter two showing a moderate degree of taper and being more desirable from a structural design point of view. It is concluded that the possibility of designing blading having high efficiency over a considerable range of operating conditions exists and can be used.

Introduction

THE theoretical aerodynamic design of the blading for a windmill of maximum efficiency, as given by Glauert, [1] using the vortex blade element theory determines the desired blade loading at any radius as a function of the ratio of tip speed to wind speed. This desired loading can be matched by using an arbitrary high lift coefficient and the correspondingly determined chord or by a lower lift coefficient and a proportionately larger chord. The designer can use this degree of freedom to deal with other factors.

Any windmill must operate in a wide range of wind conditions. All high-performance windmills that have been built have had a fairly narrow range of operating conditions at which the performance approximates the theoretical limit. Because the designer is free to choose the chord at any given operating condition and still obtain the optimum loading, the possibility of making that choice so as to obtain the optimum loading at a second operating condition exists. If such a choice does not lead to unreasonable aerodynamic or structural design requirements the result should be of importance in practical design applications. This theoretical possibility is explored in the following discussion.

The analysis is carried out using the vortex blade element theory, following Glauert. The complexity of the aerodynamic interrelations makes is necessary to carry out the exploration by numerical analysis. The numerical examples are chosen to be compatible with possible application to large wind-electric systems. The possibility of a pitch control change between design points is considered.

Performance Analysis

Consider a windmill of B blades, operating an an angular velocity Ω in a uniform wind V with an air density ρ. The blade element theory shows that the power dP developed by the elements of chord c and span dr at the radial station r is (see Fig. 1 for the wind triangle and the approach angle φ)

$$dP/dr = (\Omega r)\,(\tfrac{1}{2}\rho W^2)\,Bc\{C_L\sin\varphi - C_D\cos\varphi\} \qquad (1)$$

The differential lift and drag forces are expressed in terms of conventional nondimensional lift and profile drag coefficients and the relative wind velocity W; i.e., $dL=\tfrac{1}{2}\rho W^2\, cC_L dr$, and $dD=\tfrac{1}{2}\rho W^2 cC_D dr$. Note (from Fig. 1) that a and a' are dimensionless representations of the axial and rotational induced velocities superimposed on the primary flow by the windmill. The induction coefficients a and a', given by the vortex theory (see Glauert), are proportional to the dimensionless blade loading

$$\lambda = BcC_L/8\pi r \qquad (2)$$

with

$$a/(1-a) = \lambda\cot\varphi\csc\varphi \qquad (3)$$

and

$$a'/(1+a') = \lambda\sec\varphi \qquad (4)$$

These coefficients have the obvious interrelations

$$a'/(1+a') = [a/1-a]\tan^2\varphi \qquad (5)$$

and

$$\tan\varphi = (V/\Omega r)\,[(1-a)/(1+a')] \qquad (6)$$

By (3), the performance equation can be written

$$C_P' = \frac{(1/2\pi r)\,(dP/dr)}{\tfrac{1}{2}\rho V^3} = 4a(1-a)\left(\frac{\Omega r}{V}\right)(\tan\varphi - \epsilon) \qquad (7)$$

where $\epsilon = C_D/C_L$. From (7) it is seen that C_P' is the ratio of the power developed dP to the energy flux of the undisturbed wind through the infinitesimal annular disk area $2\pi r dr$. The relations (2) through (7) and a knowledge of the blade geometry make it possible to compute the performance of any given windmill.

Optimum Design

The relations (3) and (4) show that, within the limitations of this theory, the induction coefficients at any station depend only on parameters at that station. The problem of maximizing the power production is, thus, that of maximizing at each station the value of the product

$$F = a(1-a)\,(\tan\varphi - \epsilon) \qquad (8)$$

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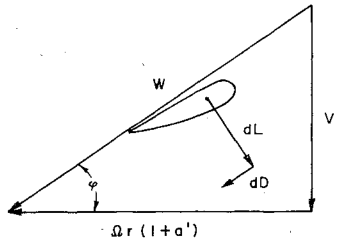

Fig. 1 Blade element relative wind triangle.

The values of a and φ corresponding to this maximum condition will be functions of the dimensionless radius $x=(\Omega r/V)$ as will the corresponding maximum value of F. In order to carry out the optimization, it is convenient to eliminate a' from (5) and (6). Thus,

$$(\Omega r/V)\tan\varphi = 1 - a\sec^2\varphi \tag{9}$$

This implicitly defines the relation between φ and a for any given value of $(\Omega r/V)$. For the moment, let us consider φ to be the independent variable. The maximum value of F then will occur at the point $dF/d\varphi=0$. If the differentiation is carried out implicitly, using (9) and $(\Omega r/V)$ then is eliminated, again using (9), it is seen that the condition for maximum F is

$$-\frac{a}{1-a}\tan^2\varphi + \frac{1-a}{a} = 2 + \frac{\epsilon\sec^2\varphi}{\tan\varphi-\epsilon} \tag{10}$$

The relation (10) expresses the relation between a and φ that must hold for any given value of ϵ; i.e., the parameter ϵ must be treated as a constant in the differentiation. Consideration of the effects of a variation in ϵ leads only to the expected result that the performance degrades steadily as ϵ increases.

Glauert's theory of optimum loading for a "perfect windmill", $\epsilon=0$, is developed readily from (10). The relation is quadratic in $(1-a)/a$, and the two solutions are

$$(1-a)/a = 1 \pm \sec\varphi \tag{11}$$

The negative sign corresponds to a negative value of a (i.e., a propeller mode of operation); so the windmill mode solution is

$$1/a = 2 + \sec\varphi \tag{12}$$

Then, from (9) and (3), the local speed ratio $x=(\Omega r)/V$ and the dimensionless loading $\lambda=(BcC_L)/8\pi r)$ are

$$x = \frac{(2\cos\varphi-1)(1+\cos\varphi)}{(2\cos\varphi+1)\sin\varphi} \tag{13}$$

$$\lambda = 1-\cos\varphi \tag{14}$$

From (7), the local power output is

$$C_P' = 4(1-a)^2(4a-1) \tag{15}$$

These relations, (12) through (15), summarize Glauert's solution ($\epsilon=0$) in a parametric form. It can be seen that as x becomes large, φ approaches zero, $a=1/3$ and $C_P'=16/27$. This limiting value of C_P' is the well-known maximum value of the power output obtained by the Froude actuator disk theory and it is frequently useful to refer to this power output as a reference in defining the efficiency of a windmill. We thus may define a local Froude efficiency as $\eta_F'=(27/16)C_P'$. An appropriate integration of (7) then can be used to define either an overall power coefficient C_P or a Froude efficiency η_F. For $\epsilon=0$, this integration was carried out by Glauert and will not be reproduced here.

For any real airfoil the profile drag is positive and $\epsilon>0$. The optimum relation (10) still holds within the limitations of the basic theory for computing the induction coefficients. For this case, (10) is still quadratic in $(1-a)/a$ and the appropriate root for the windmill mode is

$$\frac{1}{a} = 2 + \frac{\epsilon\sec^2\varphi}{2(\tan\varphi-\epsilon)} + \sec\varphi\left[\frac{\tan\varphi}{\tan\varphi-\epsilon} + \frac{\epsilon^2\sec^2\varphi}{4(\tan\varphi-\epsilon)^2}\right]^{1/2} \tag{16}$$

Once the optimum value of a is found for a given (φ, ϵ), the computation of the corresponding values of x, λ, and C_P' proceeds in exactly the same manner as in the solution for $\epsilon=0$. Because the optimum relation (16) is cumbersome, it is more convenient to leave the relations in the parametric forms (16, 9, 3, and 7) rather than the more explicit representations (13-15) for $\epsilon=0$.

It can be seen from (7) and (16) that the magnitude of the effect of the profile drag is measured by the ratio of ϵ to $\tan\varphi$. Even for a very small ϵ, the effect can be large when $\tan\varphi$ is small. As an example, Table 1 compares the optimum parameters for $\epsilon=0.01$ and the two values of $x=6.76$ ($\tan\varphi=0.10$) and 13.83 ($\tan\varphi=0.05$) with the corresponding parameters for $\epsilon=0$. The value $\epsilon=0.01$ was chosen as representing a high but attainable value of technical performance. Soaring gliders demonstrate better values but require constant maintenance. The range of φ chosen covers the operating range of current large windmill designs. For these cases $\epsilon\cot\varphi=0.1$ and 0.2, and the last factor of (7) indicates a 10% and a 20% loss. It is seen that the reduction in the local Froude efficiency is predicted almost exactly by this one factor. There is an additional loss (0.1% and 0.4%, respectively) that can be associated with the small changes in a and φ. The optimum loading λ is reduced by about 2% and 4% in the two cases. The optimum local Froude efficiency increases as x decreases reaching a maximum of about 92.5% near $x=4$. Even at $x=2.56$, the optimum local Froude efficiency is about 90%, a 4% reduction from the optimum value for $\epsilon=0$.

Dual Optimum Design

Let us apply these optimum calculations to the question of determining a blade chord length at any given radius r so that the optimum conditions will be satisfied at two different operating conditions, x_1 and x_2. For convenience, let x_1 be greater than x_2. We then can form the differences $\Delta\lambda=\lambda_2-\lambda_1$ and $\Delta\varphi=\varphi_2-\varphi_1$. Now

$$\Delta\lambda = (Bc/8\pi r)(C_{L_2}-C_{L_1}) = (Bc/8\pi r)a_0(\Delta\varphi+\tau) \tag{17}$$

where a_0 is the slope of the lift curve and τ is the pitch change between the two operating conditions and must be the same at all values of r. A positive τ increases the loading at condition x_2. Thus,

$$(Bc/8\pi r) = \Delta\lambda/[a_0(\Delta\varphi+\tau)] \tag{18}$$

and, the optimum value of λ being known at x_1 and x_2,

$$C_{L_{1,2}} = \lambda_{1,2}(8\pi r/Bc) \tag{19}$$

For negligible drag ($\epsilon=0$) and fixed pitch ($\tau=0$), a simple analytic solution is obtained if x_2 is only infinitesimally less

Table 1 Profile drag effect on optimum parameters

x	6.75		13.83	
ϵ	0.01	0	0.01	0
φ	5.71°	5.61°	2.86°	2.76°
a	0.3208	0.3328	0.3075	0.3332
η_F'	0.8948	0.9952	0.7960	0.9988
λ	0.004700	0.004789	0.001109	0.001157

Table 2 Optimum parameters for $X_1 = 5$, $X_2 = 4$, and $\epsilon = 0$

r/R		1	0.8	0.64	0.512	0.4096	0.32768
$X_1 = 5$	λ	0.00865	0.01331	0.02032	0.03066	0.04550	0.06599
	$\varphi°$	7.5400	9.3574	11.5693	14.2245	17.3502	20.9319
$X_2 = 4$	λ	0.01331	0.02032	0.03066	0.04550	0.06599	0.09291
	$\varphi°$	9.3574	11.5693	14.2245	17.3502	20.9319	24.8943
$\Delta\lambda$		0.00466	0.00701	0.01034	0.01484	0.02049	0.02692
$\Delta\varphi°$		1.8174	2.2119	2.6552	3.1257	3.5817	3.9624

Table 3 Optimum blading for $X_1 = 5$, $X_2 = 4$, $\epsilon = 0$, and $\tau = 0°$

r/R	1	0.8	0.64	0.512	0.4096	0.32768
$Bc/8\pi r$	0.0256	0.0317	0.0390	0.0475	0.0572	0.0679
$Bc/8\pi R$	0.0256	0.0254	0.0249	0.0243	0.0234	0.0223
C_{L_1}	0.337	0.420	0.522	0.646	0.795	0.971
C_{L_2}	0.519	0.641	0.787	0.958	1.153	1.368

Table 4 Optimum blading for $X_1 = 5$, $X_2 = 4$, $\epsilon = 0$, and $\tau = 1.5°$

r/R	1	0.8	0.64	0.512	0.4096	0.32768
$Bc/8\pi r$	0.0140	0.0189	0.0249	0.0321	0.0403	0.0493
$Bc/8\pi R$	0.0140	0.0151	0.0159	0.0164	0.0165	0.0161
C_{L_1}	0.615	0.705	0.816	0.956	1.128	1.339
C_{L_2}	0.947	1.076	1.232	1.418	1.636	1.885

Table 5 Optimum blading for $X_1 = 5$, $X_2 = 4$, $\epsilon = 0$, and $\tau = 2°$

r/R	1	0.8	0.64	0.512	0.4096	0.32768
$Bc/8\pi r$	0.0122	0.0166	0.0222	0.0290	0.0367	0.0451
$Bc/8\pi R$	0.0122	0.0133	0.0142	0.0148	0.0150	0.0148
C_{L_1}	0.708	0.799	0.914	1.059	1.239	1.462
C_{L_2}	1.090	1.221	1.380	1.572	1.797	2.058

than x_1. Such a design, having a double optimum at the design point x_1, should have a performance curve which is nearly optimum over an increased range of x, both above and below the design point x_1. For this case (using radian measure for φ and a_0), it follows from (14) and (18) that

$$(Bc/8\pi r) = (1/a_0)\sin\varphi \tag{20}$$

and, from (19)

$$C_L = a_0 \tan(\varphi/2) \tag{21}$$

For this example (using $a_0 = 2\pi$), C_L decreases steadily as x increases, falling from $C_L = 1.68$ at $x = 1$ ($\varphi = 30°$) to $C_L = 0.41$ at $x = 5$($\varphi = 7.54°$). The chord is constant for large x, and decreases toward the root; for example the chord at $x = 1$ is 75% of that at large x. A set of blading could be built by this rule, providing that the hub obscuration was not too small. The design tip speed $X = \Omega R/V$ also should not be too high as this would lead to large efficiency loss from the profile drag effects. This particular design, having an inverse taper, would not be attractive from the structural standpoint.

If the tip speed ratios X_1 and X_2 differ by a small, but finite, amount, the range over which the performance approximates the theoretical maximum should be expanded still further. In this case, the problem can be explored numerically. It was seen that, at least for small ϵ, the effect of profile drag on the loading parameter is small; so the examples are constructed using the theory for $\epsilon = 0$. For convenience, φ and τ are measured in degrees and a corresponding value $a_0 = 0.1$ is used ($a_0 = 5.73$ in radian measure). The values $X_1 = 5$ and $X_2 = 4$ are chosen arbitrarily. If a windmill is at a site with a mean wind speed of about 12 mph, the peak energy recovery expectation will be at wind speeds near 24 mph and the generator might logically be sized to reach full power at 30 mph. With constant angular velocity, these two speeds thus could fit the design points $X = 5$ and 4, respectively. The ratio of the available power at these two conditions is $(0.8)^3 = 0.512$ and the blading, so designed, can be expected to have high efficiency over the range in which the electric generator can have high efficiency. Table 2 shows the optimum parameters λ and φ and their differences for these two operating conditions as a function of the radial station position r/R. As a matter of computational convenience, the radial stations chosen are a geometric series with the ratio 5:4 between adjacent members. Table 3 shows the results of using the data of Table 2 to compute the chord ratios $Bc/8\pi r$ as well as C_{L_1} and C_{L_2} for a fixed blading ($\tau = 0°$). Tables 4 and 5 are similar except that the blading design of Table 4 is computed for a pitch change of $\tau = 1.5°$ and Table 5 is computed for $\tau = 2°$.

The blade chords are proportional to $Bc/8\pi R$ and it is seen from Table 3 that the fixed blades of optimum loading for $X_1 = 5$, $X_2 = 4$ have a slight inverse taper. The blade areas are somewhat larger than those used for most high-efficiency designs because the optimum values of C_{L_2} are fairly low. With pitch change, $\tau = 1.5°$ and 2°, the planform shows a progressively increased degree of taper and relatively small blade areas. Even the most extreme case, $\tau = 2°$ (which has $C_{L_2} = 2.058$ at the inner station), is within the range of modern airfoil design practice.

For small values of φ, Glauert's theoretical results take on a very simple form; namely, $\varphi = 2/(3x)$ (radians) and $\lambda = 2/(9x^2)$. This approximation is valid for surprisingly small values of x. For example at $x = 4$ these results yield $\varphi = 9.85°$ and $\lambda = 0.01389$ as compared to the more accurate values of 9.36° and 0.01331 shown in the first column of Table 2. This high tip-speed ratio approximation permits a very simple analytical formulation of the dual design problem. Setting $x_2 = 0.8x_1$, (18) and (19) become

$$\frac{Bc}{8\pi r} = \frac{0.6}{a_0 x_2 (1 + 7.5 x_2 \tau)} \tag{22}$$

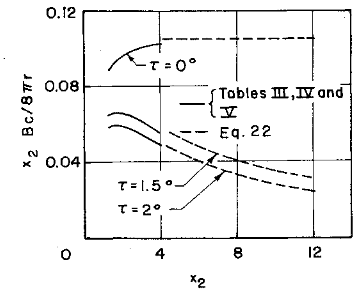

Fig. 2 Chord distribution for $x_2 = 0.8\,x_1$ ($\epsilon = 0$, $a_0 = 5.73$).

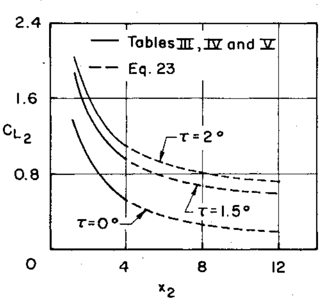

Fig. 3 C_{L_2} distribution for $x_2 = 0.8x_1$ ($\epsilon = 0, a_0 = 5.75$).

and

$$C_{L_2} = \frac{a_0}{2.7x_2}\,(1 + 7.5x_2\tau) \tag{23}$$

In these expressions, radian measure is used for a_0 and τ. If we take $\tau = \pi/120$ and $a_0 = 18/\pi$ for comparison with the data in Table 4, then for $x_2 = 4$, (22) and (23) show $(Bc/8\pi r) = 0.0147$ and $C_{L_2} = 0.944$. The more accurately computed values of 0.0140 and 0.947 are shown in the first column of Table 4. The error in the calculation varies inversely as the square of x.

It is of interest to note that the data of Tables 3-5 can be transformed easily to any other design tip speed as long as the new X_1, X_2 are in the ratio of 5 to 4. One need only multiply through the rows r/R and $Bc/8\pi R$ by the ratio of 4 to the new X_2. For a design with $X_2 = 10$ and $\tau = 1.5°$, Table 4 then shows the blading to have a maximum chord at the station $r/R = 0.4 \times 0.4096$ at which $Bc/8\pi r = 0.0403$ so $Bc/8\pi R = 0.0066$. The design lift coefficients at the design point $X_2 = 10$ are $C_{L_2} = 1.636$ at the maximum chord point and $C_{L_2} = 0.645$ at the tip. Such a design shows a taper ratio of almost two to one and a very slender, high aspect ratio blade. The tip chord (for $B = 2$) would be only 2.75 ft for $R = 62.5$ ft. Interpolation from Table 1 indicates that such a design (for $\epsilon = 0.01$) would have about 3% too high a tip loading and that the tip chord should be reduced by about 5% with a compensating 2% increase in the tip lift coefficient. Such a design would have a substantial efficiency loss because of profile drag and would also be quite sensitive to the twist program; one degree of improper twist would shift the loading by 15%. The design thus outlined for $X_2 = 10$, $\epsilon = 0$, $\tau = 1.5°$ would require an aerodynamic twist of 7.20° between the maximum chord station and the tip.

The principal results of these dual optimum design calculations for $x_2 = 0.8x_1$ are presented graphically in Figs. 2 and 3. The blade chord distribution (Fig. 2) is shown as the product $x_2\,(Bc/8\pi r)$ because this quantity shows directly the required blade shape but is a function of only local parameters. It does not require a specification of the design tip-speed ratio. Because $x_2 = (r/R)X_2$ and $X_2 = 4$ in Tables 3-5, the plotted ordinate is four times the value of $(Bc/8\pi R)$ presented in the tables.

If, as was assumed earlier, the design condition X_2 represents the wind speed at which the maximum energy recovery is reached, then C_{L_2} (Fig. 3) is close to the maximum C_L requirement for the blades and, thus, is an appropriate criterion for airfoil choice. Once the airfoils are chosen, the blade twist is determined. The aerodynamic requirements imposed in the example displayed, $x_2 = 0.8x_1$, are not extreme except for low values of x_2, although the most extreme case ($\tau = 2°$) approaches the current technological limit at the last point ($x_2 = 1.31$) computed.

The data on the two figures are closely interrelated. The product of the two ordinates is $x_2\lambda_2$ and Glauert's text shows this product graphically as well as in tabular form. It easily is verified that for each of the three cases presented, the product of C_{L_2} and $x_2\,(Bc/8\pi r)$, for a given x_2, is the same and is in accord with Glauert's results. Similar design charts could be constructed for other values of the ratio (x_2/x_1).

Conclusion and Discussion

The analysis presented leads to the obvious conclusion that an aerodynamic objective of designing windmill blading that can have a high efficiency over a fairly wide range of operating conditions is feasible and that the resulting design can, by a suitable pitch change program, be compatible with reasonable structural desires. The data of Tables 3-5 and of Figs. 2 and 3 are based on Glauert's optimum analysis ($\epsilon = 0$); however, the more complete analysis for $\epsilon > 0$ shows that the effects, at least for $\epsilon = 0.01$, would be very small, a few percent reduction in chord and a somewhat smaller increase in C_L. The effect would be largest near the tip of the blade, and the resulting slight increase in taper is in the sense to improve structural design conditions. To reproduce the calculations for any assigned value (or values) of ϵ is a straightforward computing problem and would be a logical step in any preliminary design analysis.

The overall accuracy of these design calculations is of course limited by the precision of the vortex blade element theory; however, design experience has shown the theory to be of good accuracy for preliminary design purposes. It neglects local blade tip losses, overlooks the blade root and hub obscuration problems, assumes an average value of the annulus for the induction coefficients, and neglects the effect of profile drag on the induction coefficients. These approximations are of small enough effect that the general conclusions herein can be supported even though based purely on analysis. I look forward to the time when these conclusions can be brought to the test of physical experiment.

Reference

[1] Glauert, H., "Airplane Propellers," *Aerodynamic Theory*, edited by W.F. Durand, Julius Springer, Berlin, 1935, pp. 324-332.